Generalized Relative Order of Functions of Several Complex Variable Analytic in the Unit Poly Disc

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Abstract: Throughout in this paper we consider generalized relative order of a function of several complex variable analytic in the unit poly disc with respect to an entire function and prove several theorems.

Key words: Analytic function, entire function, generalized relative order, poly disc, property (R).

Introduction, Definition, and Notation.

In the unit disc $U : \{z : |z| < 1\}$ a function f analytic, is said to be of finite Nevanlinna order [4] (Juneja and Kapoor 1985) if there exists a number μ such that Nevanlinna characteristic function $T(r, f)$ of defined by

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta$$

satisfies

$$T(r,f) = (1-r)^{-\mu}$$

for all r in $0 < r_0(\mu) < r < 1$

Thus the Nevanlinna order $\rho(f)$ of f is given by $\rho(f) = \lim_{r\to 1} sup \frac{\log T(r,f)}{\log(1-r)^{-1}}$

Introduced the idea of relative order of an entire function in [1] Banerjee and Dutta which as follows-

***Definition* 1**. If in U, f is an analytic and g be entire , then the relative order of f with respect to g, denoted by

$$\rho_g(f) = inf\left\{\begin{matrix}\mu > 0 :\ T_f(r) < T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right], \\ \forall\ 0 < r_0(\mu) < r < 1\end{matrix}\right\}$$

When $g(z) = \exp z$ then the definition 1 coincides with definition of Nevanlinna order of f.

***Definition* 2**. Let $f(z_1, z_2)$ be non constant analytic function of two complex variables z_1 *and* z_2 holomorphic in the closed poly disc $P : \{(z_1, z_2) : |z_j| \le 1; j = 1,2\}$ and $g(z_1, z_2)$ be an entire function then relative order of f with respect to g is defined by

$$\rho_g(f) = inf\left\{\begin{matrix}\mu > 0 : F(r_1, r_2) \\ < G\left(\frac{1}{(1-r_1)^{\mu}}, \frac{1}{(1-r_2)^{\mu}}\right), \forall\ 0 < r_0(\mu) < r_1, r_2 < 1\end{matrix}\right\}$$

Dutta [5] introduced the following definition.

***Definition* 3**. Let two entire function $f(z_1, z_2, \ldots, z_n)$ and $g(z_1, z_2, \ldots, z_n)$ of n complex variables with maximum modulus functions $F(r_1, r_2, \ldots, r_n)$ and $G(r_1, r_2, \ldots, r_n)$ respectively, then relative order of f with respect to g defined by

$$\rho_g(f) = inf\{\mu > 0 : F(r_1, r_2, \dots, r_n) < G(r_1^{\mu}, r_2^{\mu}, \dots\dots, r_n^{\mu}) for\ r_i \geq R(\mu); i = 1,2,3, \dots n\}$$

Dutta [6] introduced in this paper the following definition.

***Definition* 4**. Let

$$f(z_1, z_2, \dots\dots, z_n) = \sum_{m_1, m_2, \dots m_n = 0}^{\infty} C_{m_1 m_2 \dots\dots m_n} z_1^{m_1} z_2^{m_2} \dots . z_n^{m_n}$$

be a function of n complex variables $z_1, z_2, \dots\dots, z_n$ holomorphic in the unit poly disc

$P : \{(z_1, z_2) : |z_j| \leq 1; j = 1,2\}$ and the maximum modulus,

$$F(r_1, r_2, \dots, r_n) = max\{|F(z_1, z_2, \dots\dots, z_n)| : |z_j| \leq r_j ; j = 1,2, \dots\dots, n\}$$

The order ρ and lower order λ are defined as

$$\rho = \lim_{r_1, r_2, \dots, r_n \to 1} sup \frac{\log \log F(r_1, r_2, \dots, r_n)}{-\log(1 - r_1)(1 - r_2) \dots . (1 - r_n)}$$

and

$$\lambda = \lim_{r_1, r_2, \dots, r_n \to 1} inf \frac{\log \log F(r_1, r_2, \dots, r_n)}{-\log(1 - r_1)(1 - r_2) \dots . (1 - r_n)}$$

***Definition* 5**. In the closed unit poly disc , $f(z_1, z_2, \dots\dots, z_n)$ be a non constant analytic of several complex variables $P : \{(z_1, z_2, \dots\dots, z_n) : |z_j| \leq 1;\ j = 1,2, \dots\dots, n\}$ and $g(z_1, z_2, \dots\dots, z_n)$ be an entire function then relative order of f with respect to g denoted by

$$\rho_g(f) = inf \left\{ \begin{array}{c} \mu > 0 : F(r_1, r_2, \dots, r_n) \\ < G\left(\frac{1}{(1 - r_1)^{\mu}}, \frac{1}{(1 - r_2)^{\mu}}, \dots\dots, \frac{1}{(1 - r_n)^{\mu}} \right) \forall, \\ 0 < r_0(\mu) < r_1, r_2, \dots, r_n < 1 \end{array} \right\}$$

where $G(r_1, r_2, \dots, r_n) = max\{|g(z_1, z_2, \dots\dots, z_n)| : |z_j| = r_j ;\ j = 1,2, \dots\dots, n\}$.

when $g(z_1, z_2, \dots\dots, z_n) = e^{z_1 z_2 \dots . z_n}$ then definition 5 coincides definition 4 and if $n = 2$ then coincide with definition 2.

***Definition* 6**. An entire function $g(z_1, z_2, \dots\dots, z_n)$ is said to have the property (R) if for any $\sigma > 1, \lambda > 0$ and for all r_i sufficiently close to 1; $i = 1,2, \dots\dots, n$

$$G\left[\left(\frac{1}{(1 - r_1)^{\lambda_1}}, \frac{1}{(1 - r_2)^{\lambda_2}}, \dots\dots, \frac{1}{(1 - r_n)^{\lambda_n}} \right)\right]^2 < G\left(\frac{1}{((1 - r_1)^{\lambda})^{\sigma}}, \frac{1}{((1 - r_2)^{\lambda})^{\sigma}} \dots\dots, \frac{1}{((1 - r_n)^{\lambda})^{\sigma}} \right)$$

If $g(z_1, z_2, \dots\dots, z_n) = z_1 z_2 \dots . z_n$ has not property of (R) but $g(z_1, z_2, \dots\dots, z_n) = e^{z_1 z_2 \dots . z_n}$ has the property of (R) those in available in [3] *and* [4]. Dutta and Jerin introduced the idea of generalized relative order.

***Definition* 7**. Introduced the idea of generalized relative order in [1] Nevanlinna's characteristic function of f is denoted by $T_f(r_1, r_2, \dots, r_n)$. The relative generalized Nevanlinna order of an analytic function f in unit U with respect to another entire function g are defined by

$$\rho_g^P(f(r_1, r_2, \dots, r_n)) = \lim_{r_1, r_2, \dots, r_n \to 1} sup \frac{\log^{[P]} T_g^{-1} T_f(r_1, r_2, \dots, r_n)}{-\log(1 - r_1)(1 - r_2) \dots . (1 - r_n)}$$

***Lemma* 1**. [6] If entire function f have the property of (R). Then for any positive integer n and for all $\sigma > 1, \lambda > 0$,

$$G\left[\left(\frac{1}{(1-r_1)^{\lambda_1}},\frac{1}{(1-r_2)^{\lambda_2}},\dots,\frac{1}{(1-r_n)^{\lambda_n}}\right)\right]^n$$

and

$$< G\left(\frac{1}{((1-r_1)^{\lambda})^{\sigma}},\frac{1}{((1-r_2)^{\lambda})^{\sigma}},\dots,\frac{1}{((1-r_n)^{\lambda})^{\sigma}}\right)$$

Where r_i , $0 < r_i < 1, i = 1,2,\dots,n$.

***Lemma* 2**. [6] If g is entire function then

$$T_g\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)$$

$$\leq \log G\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)$$

$$\leq 3T_g\left(\frac{2}{(1-r_1)},\frac{2}{(1-r_2)},\dots,\frac{2}{(1-r_n)}\right)$$

For all $r_1,r_2,\dots,r_n$, $0 < r_1,r_2,\dots,r_n < 1$.

***Theorem* 1**. Let in U , f be analytic of generalized relative order $\rho_g^P(f)$ where g is entire . Let $\varepsilon > 0$ be arbitrary

$$T_f(r_1,r_2,\dots,r_n) = O\left(\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_g^P f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right)$$

holds for all $r_1,r_2,\dots,r_n$, $0 < r_1,r_2,\dots,r_n < 1$, conversely if for an analytic f in U and entire g having the property (R)

$$T_f(r_1,r_2,\dots,r_n) = O\left(\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)\right)$$

holds for all $r_1,r_2,\dots,r_n$, $0 < r_1,r_2,\dots,r_n < 1$, sufficiently closed to 1, and

$$T_f(r_1,r_2,\dots,r_n) = O\left(\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k-\varepsilon}\right)\right)$$

does not holds for all $r_1,r_2,\dots,r_n$, $0 < r_1,r_2,\dots,r_n < 1$, then

$$k = \rho_g^P f(r_1,r_2,\dots,r_n)$$

***Proof*.** From the definition of generalized relative order, we have

$$T_f(r_1,r_2,\dots,r_n) = T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_g^P f(r_1,r_2,\dots,r_n)+\varepsilon}\right)$$

for $0 < r_1,r_2,\dots,r_n < 1$, or

$$T_f(r_1,r_2,\dots,r_n) = \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_g^P f(r_1,r_2,\dots,r_n)+\varepsilon}\right)$$

$0 < r_1,r_2,\dots,r_n < 1$, by lemma 2 so

$$T_f(r_1,r_2,\dots,r_n) = O\left(\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_g^P f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right)$$

conversely, if

$$T_f(r_1,r_2,\dots,r_n) =$$

$$O\left(\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)\right)$$

holds for all $r_1,r_2,\dots,r_n$, $0<r_1,r_2,\dots,r_n<1$, then

$$T_f(r_1,r_2,\dots,r_n)$$

$$=[\alpha]\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right),\alpha$$

>1,

$$=\frac{1}{3}\log\left[G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)\right]^{[3\alpha]}$$

$$\leq\frac{1}{3}\log G\left(\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)^{\sigma}\right)$$

By lemma 1 for any $\sigma>1$,

$$\leq T_g\left[2\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)\right]^{\sigma}$$

by lemma 2 since

$$\log T_g^{-1}\,T_f(r_1,r_2,\dots,r_n)$$

$$\leq\log 2+\log\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}\right)^{\sigma}$$

$$\leq\sigma exp^{[p-2]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}+O(1)$$

Since,

$$\log^{[2]}T_g^{-1}\,T_f(r_1,r_2,\dots,r_n)$$

$$\leq exp^{[p-3]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k+\varepsilon}+O(1)$$

So

(a)

$$\lim_{r_1,r_2,\dots,r_n\to 1-}sup\frac{\log^{[P]}T_g^{-1}T_f(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots.(1-r_n)}\leq k+\varepsilon$$

Since $\varepsilon>0$, we have

$$\lim_{r_1,r_2,\dots,r_n\to 1-}sup\frac{\log^{[P]}T_g^{-1}\,T_f(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots.(1-r_n)}\leq k$$

$r_1,r_2,\dots,r_n$ tending to $1-$ then there exist a sequence $\{r_n\}$ for which

$$T_f(r_1,r_2,\dots,r_n)$$

$$=\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k-\varepsilon}\right)$$

$$\geq T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{k-\varepsilon}\right)$$

by lemma 2 and so

(b)

$$\lim_{r_1,r_2,\dots,r_n\to 1-}sup\frac{\log^{[P]}T_g^{-1}T_f(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots.(1-r_n)}\geq k-\varepsilon$$

$for\ r = r_1, r_2, \dots, r_n \to 1-$ combining (a) and (b) we obtain $k = \rho_g^p\big(f(r_1, r_2, \dots, r_n)\big)$

hence proof the theorem.

SUM AND PRODUCT THEOREM

***Theorem* 2.** In the unit disc U, having f_1 and f_2 of generalized relative orders $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)$ respectively, where g is entire having the property (R) then

(i) $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n) + f_2(r_1, r_2, \dots, r_n)\big) \le max\{\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big), \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)\}$

(ii) $\rho_g^p\big(f_1(r_1, r_2, \dots, r_n). f_2(r_1, r_2, \dots, r_n)\big) \le max\{\rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big), \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)\}$

the some inequality holds for quotients the equality holds in $(ii)\ if\, \rho_g^p\big(f_1(r_1, r_2, \dots, r_n)\big) \ne \rho_g^p\big(f_2(r_1, r_2, \dots, r_n)\big)$.

***Proof*.** Let $\rho_1 = \rho_g^{[p]}\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_2 = \rho_g^{[p]}\big(f_2(r_1, r_2, \dots, r_n)\big)$ and $\rho_1 \le \rho_2$. We assume that $\rho_g^{[p]}\big(f_1(r_1, r_2, \dots, r_n)\big)$ and $\rho_g^{[p]}\big(f_2(r_1, r_2, \dots, r_n)\big)$ both are finite because if one of them or both are infinite inequality are evident for arbitrary $\varepsilon > 0$ and for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1 we have

$$T_{f_1}(r_1, r_2, \dots, r_n)$$

$$< T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right)$$

and

$$T_{f_2}(r_1, r_2, \dots, r_n)$$

$$< T_g\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

Using lemma 2 for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1

$$T_{f_1 \ne f_2}(r_1, r_2, \dots, r_n)$$

$$\le T_{f_1}(r_1, r_2, \dots, r_n) \pm T_{f_2}(r_1, r_2, \dots, r_n) + O(1)$$

$$\le \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_1+\varepsilon}\right)$$

$$+ \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right) + O(1)$$

$$\le 3 \log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)$$

$$= \frac{1}{3} \log\left[G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)\right]^9$$

$$\leq \frac{1}{3}\log G\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}$$

by lemma 1, for any $\sigma > 1$

$$\leq T_g\left(2\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}\right)$$

by lemma 2, since

$$T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n) \leq$$

$$\log 2 + \log\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon}\right)^{\sigma}$$

$$\leq \sigma exp^{[p-2]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon} + O(1)$$

$$\log^{[2]} T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n) \leq exp^{[p-3]}\left(\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\dots,\frac{1}{(1-r_n)}\right)^{\rho_2+\varepsilon} + O(1)$$

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)+f_2(r_1,r_2,\dots,r_n)\right)$$

$$= \lim_{r_1,r_2,\dots,r_n\to 1-} sup\frac{\log^{[P]} T_g^{-1}T_{f_1\neq f_2}(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots(1-r_n)} \leq \rho_2+\varepsilon$$

since $\varepsilon > 0$ is arbitrary,

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)+f_2(r_1,r_2,\dots,r_n)\right) \leq \rho_2$$

$$\leq max\left\{\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)\right\}$$

which proves (i), for (ii), since

$$T_{f_1\cdot f_2}(r_1,r_2,\dots,r_n) \leq T_{f_1}(r_1,r_2,\dots,r_n) + T_{f_2}(r_1,r_2,\dots,r_n)$$

we obtain similarly as above

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n).f_2(r_1,r_2,\dots,r_n)\right)$$

$$\leq max\left\{\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)\right\}$$

Let $f = f_1f_2$ and

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) < \rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

Then applying (ii), we have

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) \leq \rho_g^{[p]}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

again since $f_2 = \frac{f}{f_1}$, applying the first part of (ii), we have

$$\rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

$$\leq max\left\{\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right),\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right)\right\}$$

since

$$\rho_g^{[p]}\left(f_1(r_1,r_2,\dots,r_n)\right) < \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

we have

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) \leq \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

$$= max\{\rho_g^{p}\left(f_1(r_1,r_2,\dots,r_n)\right),\rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)\}$$

when

$$\rho_g^{p}\left(f_1(r_1,r_2,\dots,r_n)\right) \neq \rho_g^{p}\left(f_2(r_1,r_2,\dots,r_n)\right)$$

this prove the theorem.

RELATIVE ORDER WITH RESPECT TO THE DERIVATIVE OF AN ENTIRE FUNCTIONS

***Theorem* 3**. In the unit disc, f is analytic function and g be transcendental entire having the property (R), then

$$\rho_g^{[p]}\left(f(r_1, r_2, \dots, r_n)\right) = \rho_{g'}^{[p]}\left(f(r_1, r_2, \dots, r_n)\right)$$

where g' denotes the derivative of g . To prove the theorem we require the following lemmas.

***Lemma* 3**. [1] If g be transcendental entire, then for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1 for any $\lambda > 0$

$$T_{g'}\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)$$

$$\leq 2T_g\left(2\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)\right)$$

$$+O\left(T_g\left(2\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)\right)\right)$$

***Lemma* 4**. [1] If g be transcendental entire, then for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1 for any $\lambda > 0$

$$T_g\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)$$

$$\leq \alpha_0\left[T_{g'}\left(2\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)\right)\right]$$

$$+\log\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)$$

Where α_0 is constant which is only dependent on $g(0)$.

PROOF OF THE THEOREM

Proof. We obtain for $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1 from the lemma 3 and lemma 4.

(c)

$$T_{g'}\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)$$

$$< [c]T_g\left(2\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)\right)$$

and

(d)

$$T_g\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)$$

$$< [c_0]T_{g'}\left(2\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)\right)$$

Where c_0 and $\lambda > 0$ be any number from the definition of $\rho_{g'}^{[p]}\left(f(r_1, r_2, \dots, r_n)\right)$, we get for any arbitrary $\varepsilon > 0$

$$T_f(r_1, r_2, \dots, r_n) < T_{g'}$$

$$\left(exp^{[p-1]}\left(\frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \dots, \frac{1}{(1-r_n)^\lambda}\right)^{\rho_{g'} f(r_1, r_2, \dots, r_n)+\varepsilon}\right)$$

for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, from (c) and by lemma 1 and lemma 2

for all $r_1, r_2, \dots, r_n, 0 < r_1, r_2, \dots, r_n < 1$, sufficiently close to 1

$$T_f(r_1, r_2, \dots, r_n) <$$

$$[c]T_g\left(2exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_{g'}^{[p]} f(r_1, r_2, \dots, r_n)+\varepsilon}\right)$$

$$\leq$$

$$[c]\log G\left(2exp^{[p-1]}\left(\frac{1}{(1-r_1)}, \frac{1}{(1-r_2)}, \dots, \frac{1}{(1-r_n)}\right)^{\rho_{g'}^{[p]} f(r_1, r_2, \dots, r_n)+\varepsilon}\right)$$

$$= \frac{1}{3}\log\left[G\left(2exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right]^{3[c]}$$

$$\le \frac{1}{3}\log\left(G\left(2exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right)^{\sigma}$$

For any $\sigma > 1$

$$\le T_g\left(2^{\sigma+1}\left(exp^{[p-1]}\begin{pmatrix}\frac{1}{(1-r_1)},\frac{1}{(1-r_2)},\\ \dots,\frac{1}{(1-r_n)}\end{pmatrix}^{\rho_{g'}^{[p]}f(r_1,r_2,\dots,r_n)+\varepsilon}\right)\right)^{\sigma}$$

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$$

$$= \lim_{r_1,r_2,\dots,r_n\to 1-} sup \frac{\log^{[P]} T_g^{-1} T_f(r_1,r_2,\dots,r_n)}{-\log(1-r_1)(1-r_2)\dots.(1-r_n)}$$

$$\le \rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)+\varepsilon$$

since $\varepsilon > 0$ is arbitrary, so

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) \le \rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$$

from (d) we obtain similarly,

$$\rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) \le \rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$$

so

$$\rho_g^{[p]}\left(f(r_1,r_2,\dots,r_n)\right) = \rho_{g'}^{[p]}\left(f(r_1,r_2,\dots,r_n)\right)$$

Hence prove the theorem.

Acknowledgement. The author is thankful to the referee for his helpful suggestion and guidance for the preparation of this paper.

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